ACE QUIZBOWL CAMP 2017 - MATH (Instructor: Joe Czupryn)

Mathematicians

Leonhard Euler (Swiss)

- Namesake of the number e
- Solved the Seven Bridges of Konigsberg problem, which is considered the start of graph theory (led to the path visiting each edge of a graph being named for him)
- His totient function (denoted phi) takes positive integer inputs and outputs the number of positive integers smaller than or equal to the input that are relatively prime to the input (including itself)
- His formula/characteristic states Vertices – Edges + Faces = 2 (true for all convex polyhedra)
- Names the identity $e^{i\pi} + 1 = 0$
- Names the formula that $e^{ix} = \cos(x) + i\, sin(x)$
- With Mascheroni, names a constant measuring the difference between the harmonic series and the natural log
- Names the first-order Runge-Kutta method used to solved ordinary differential equations
- Names a line passing through the orthocenter, circumcenter, and centroid of a triangle
- First to provide solution of pi-squared over six as solution to the Basel problem

Bernhard Riemann (German)

- Names the sums used to approximate definite integrals
- Namesake hypothesis concerns the distribution of primes and relates to the location of the nontrivial zeros of his zeta function (posited they are on the critical line, which means they have real part one-half)
- Names an Open Mapping Theorem that concerns mapping onto the open unit disk
- Developed elliptic geometry
- Has a namesake type of geometry that studies his namesake manifold and metric
- The integration named after him is contrasted with Lesbesgue integration

Pierre de Fermat (French)

- His numbers are of the form $2^{2^n} + 1$
- His principle of least time states that light rays travel to minimize the time taken
- **Fermat's Last Theorem**
    - States that if n is greater than two, there are no integer solutions to $a^n + b^n + c^n$
    - Proven by Andrew Wiles
    - Proof of it relied on the Taniyama-Shimura Conjecture
    - Ken Ribet proved the Epsilon Conjecture by proving this could be translated into a non-modular elliptic curve
    - Can be proven by showing all elliptic curves are modular forms
    - Sophie Germain worked on proving it
    - Other buzz words: Frey curve, Iwasawa theory
- **Fermat's Little Theorem**
    - States that for a prime p and a natural number a, $a^p \equiv a\ (mod\ p)$
    - Fails for Carmichael numbers
    - Basis for RSA encryption (CAREFUL with context, plenty of other questions will mention RSA)
    - Can be generalized using Euler's totient function
    - Wilson's Theorem is a corollary of it
    - Its converse is the basis of the Pratt certificate

Euclid (Greek)

- Considered "Father of Geometry," penned *Elements* (contained his five postulates, including the fifth one known as the parallel postulate)
- Names an algorithm that repeatedly takes the difference between a pair of numbers to find the GCF (greatest common factor)
- Proved that there are infinitely many primes
- His namesake orchard consists of one-dimensional trees planted at lattice points

Blaise Pascal (French)

- Names a triangle that contains binomial coefficients
- His correspondence with Fermat led to the development of probability theory
- Names a rule that can be used to add two combinations together to get a third combination
- Other non-math buzz words: *Provincial Letters* (defended Antoine Arnauld and attacked the Jesuits), *Pensees*, namesake wager concerning the existence of God

Carl Friedrich Gauss (German)

- The normal distribution is sometimes named after him (Gaussian distribution)
- Proved the law of quadratic reciprocity
- As a child, quickly summed the numbers from 1 to 100
- Created the method of least-squares approximation (to predict the location of Ceres)
- Requested a 17-sided polygon be inscribed on his gravestone after he proved it could be constructed
- His elimination method is used in solving systems of linear equations via matrices
- With Jordan, he names a method of getting a matrix into reduced row echelon form
- With Bonnet, he names a theorem relating a manifold's curvature to its Euler characteristic

Georg Cantor (German)

- Used his diagonalization argument to show that the reals are not countably
- Names a set that is created by repeatedly removing the middle thirds of line segments (this is an example of a fractal)
- Names a function related to his namesake set that when graphed is referred to as the Devil's Staircase
- Proposed the continuum hypothesis, which became the first of Hilbert's 23 problems (states that there is no set with a cardinality between the set of the natural numbers and the set of reals)
- His theorem states that the cardinality of a set is less than that of its power set
- He developed the idea of transfinite numbers (numbers larger than all finite numbers but not necessarily infinite)

Augustin-Louis Cauchy (French)

- With Schwarz, names an inequality stating that the dot product of two vectors must be less than or equal to the product of their magnitudes
- Namesake integral theorem states that the path integral of a holomorphic function over a closed path is zero
- Names sequences whose terms are arbitrarily close to one another
- Sylow's first theorem is a generalization of a theorem named for him
- He and Riemann name a set of differential equations that any complex differentiable function will satisfy
- Other non-math buzz words: Generalized Hooke's Law to three dimensions

David Hilbert (German)

- Presented a list of 23 important unsolved problems in mathematics in 1900

August Mobius (German)

- Namesake strip is a non-orientable surface with only one side (formed by taking a strip of paper, twisting it, and taping the ends)
- Names transformation mapping the Riemann sphere onto itself (used in hyperbolic geometry)

Sets of Numbers

Complex Numbers
- Set of numbers that can be written in the form a + bi
- These numbers can be plotted on an Argand diagram
- Holomorphic functions have these numbers as inputs
- Quaternions are an extension of these numbers in four dimensions
- DeMoivre's Theorem is useful when raising these numbers to an exponent
- Taking the absolute value of one of these numbers is known as the modulus
- Gaussian integers are a subset of these numbers (where both the real and imaginary part are integers)
- Mobius transforms take this set to itself

Real Numbers
- Represented R
- Can be constructed using Dedekind cuts (LISTEN TO THE CONTEXT!)
- Shown to be uncountably infinite by Cantor's diagonalization argument / subject of Cantor's continuum hypothesis that there is no set with a cardinality between these and the integers
- Can be constructed using Cauchy sequences (always converge in this set)
- Satisfy the least upper bound property
- Only complete, ordered Archimedean field

Irrational Numbers
- Numbers that cannot be represented as a ratio of integers / have a decimal that neither terminates nor repeats

Rational Numbers
- Represented Q
- Numbers that can be represented as a ratio of integers
- Farey sequences consist of these numbers
- Extended with the use of Dedekind cuts / acted upon by Dedekind cuts (LISTEN TO THE CONTEXT!)
- Generalized by the p-adics
- Smallest field with characteristic zero

Integers
- Represented Z
- Contains the natural numbers, their negatives, and zero
- Diophantine equations can have solutions only from this set
- Gaussian numbers have both their real and imaginary parts from this set
- Eisenstein and Gauss name "types of these numbers" (both are actually complex numbers but are called "integers")
- Shown to be countably infinite / subject of Cantor's continuum hypothesis that there is no set with a cardinality between these and the reals

Natural Numbers
- Represented N
- Contains all positive integers (and sometimes zero depending on definition)

Transcendental Numbers
- Complement of the set of algebraic numbers
- Any number that is not a root of a polynomial with rational coefficients

Algebraic Numbers
- Complement of the set of transcendental numbers
- Any number that is a root of a polynomial with rational coefficients

Prime Numbers

- Numbers only divisible by one and themselves
- Can be found using the Sieve of Eratosthenes
- Euclid proved there are infinitely many of these
- Goldbach's Conjecture states that every even natural number larger than two can be written as the sum of two of these numbers
- The dark spaces on an Ulam spiral represent these numbers
- There are approximately $\frac{n}{\ln(n)}$ of these numbers less than n
- These numbers are the arguments in the logarithms in the von Mangoldt function
- The Miller-Rabin algorithm determines is a number is of this type
- Numbers having this property is the subject of Wilson's Theorem
- Despite passing the test provided by Fermat's Little Theorem, Carmichael numbers are not these numbers
- The Green-Tao Theorem concerns sequences of these numbers
- Types of primes:
  - **Mersenne Primes** – primes of the form $2^p - 1$ (let p represent a prime number and this is always prime)
  - **Fermat Primes** – primes of the form $2^{2^n} + 1$ (only five known and that is believed to be all of them)
  - Germain
  - Wall-Sun-Sun

Fibonacci Numbers

- Numbers belonging to the sequence (0,) 1, 1, 2, 3, 5, 8, 13, 21 / found by adding the previous two
- Zeckendorf's Theorem states any natural number can be written uniquely as a sum of nonconsecutive ones of these numbers
- Binet's formula is a closed form expression of these numbers (CAREFUL with wording)
- These numbers are the subject of Carmichael's Theorem
- The limit of the ratio of consecutive ones of these numbers approaches the golden ratio
- The shallow diagonals of Pascal's triangle sum to these numbers
- With modular arithmetic, these numbers give rise to Pisano periods
- Introduced in their namesake's book *Liber Abaci*
- Wall-Sun-Sun primes are defined using these numbers

Perfect Squares

- Products of multiplying an integer by itself
- Adding two consecutive triangular numbers yields one of these
- The sum of the infinite series of reciprocals of these numbers equals pi over six
- The sum of the first n of these is $\frac{n\,(n+1)\,(2n+1)}{6}$
- Lagrange proved that all positive integers can be written as the sum of at most four of these numbers
- Congruent to zero or one modulo four, which is basis for (Gauss's) quadratic reciprocity

Perfect Numbers

- Numbers that equal the sum of their proper divisors
- Include 6, 28, and 496 (no known odd examples)
- In one-to-one correspondence with the Mersenne primes (subject of the Euler-Euclid Conjecture)
- The aliquot sequence is constant for them

Everything Else

Matrices

- Rectangular arrays of numbers
- The determinant is an operation on these
- Gaussian elimination and Gauss-Jordan elimination can be performed on these
- These can be put into reduced-row-echelon form
- The trace and rank of these can be calculated
- The characteristic polynomial for these can be used to find their eigenvalues
- The Jacobian type of these contain terms that have had the derivative taken
- Multiplication of these is non-commutative and to multiply them, the number of columns of the first needs to match the number of rows of the second
- The Hermitian type of these contains complex conjugates
- Upper and lower triangular are a type of these

Determinant

- Matrix operation that for a 2x2 matrix is equal to ad – bc (and only exists for square matrices)
- This operation can be done using cofactor expansion or expansion by minors
- If this value is nonzero, a matrix is invertible
- Cramer's rule uses a quotient of these values to solve linear systems
- The product of all eigenvalues equals this value
- This operation on the vectors of a parallelogram will yield its area
- The Wronskian is a type of this operation
- This operation is used to find the characteristic polynomial

Polynomials

- Functions consisting of the sum/difference of variables with coefficients and positive integer exponents
- No general solution for these that are fifth degree or higher (proved by Abel and Ruffini – AKA Abel-Ruffini Theorem)
- The Stone-Weierstrass Theorem (or sometimes just called the Weierstrass Theorem) uses these to approximate
- Cardano's Theorem is used to solve a type of these
- Truncating a Taylor series will give you one of these
- Splines are created from these and used for interpolation
- The characteristic one of these are used to find the eigenvalues
- Descartes' Rule of Signs applies to these
- This being irreducible over the rationals is the subject of Eisenstein's Theorem
- These are the subject of Hilbert's Basis Theorem
- Can be evaluated using Horner's Rule
- A Grobner basis is related to these

Continuity/Continuous

- Property of functions that have no holes or breaks (can trace them without lifting your pencil)
- Every differentiable function has this property (but the converse does not hold – ex. Weierstrass function, which has this property)
- Formally defined as having this property for a point c if $\lim_{x \to c} f(x) = f(c)$
- Lipschitz names a form of this property
- Functions with this property are subject to the Intermediate Value Theorem and the Extreme Value Theorem
- If for every epsilon, there exists a delta such that |x-y| is less than delta implies |f(x) – f(y)| is less than epsilon, then a function has the "uniform" type of this
- Probability functions of this type are contrasted with discrete distributions and have a probability of zero at any individual point

Derivative/Differentiable

- Inverse operation of integration
- Gives the rate of change of a variable
- For composition of functions, one uses the chain rule for this process
- This operation is applied when using L'Hopital's Rule for limits
- Subject of the Mean Value Theorem and Rolle's Theorem
- The Jacobian is a matrix in which the entries have had this operation done on them
- The Weierstrass function is continuous everywhere but has this property nowhere
- Satisfying the Cauchy-Riemann equations is related to this property
- This operation is formally defined as $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
- This operation is applied repeatedly to find the Taylor series
- This operation is related to Christoffel symbols

Integration

- Inverse operation of differentiation
- Process by which the area under a curve is found
- Can be done by parts, with partial fractions, or using u-substitution
- Forms named after Riemann, Lesbesgue, and Darboux
- Can be approximated using Riemann sums (left hand, right hand, midpoint, trapezoidal, and Simpson's)
- Fubini's theorem deals with changing the order of this process when it happens multiple times
- In spherical coordinates, the Jacobian determinant is needed for this process
- This process is the subject of Green's Theorem and Stokes' Theorem

Factorial

- Denoted by an exclamation point
- For an input n, finds the product of the first n positive integers
- Used to compute permutations and combinations
- The gamma function is a generalization of this function (for all complex numbers)
- The Pochhammer symbol represents this function's rising or falling type
- Function found in the denominators of Taylor series
- This function appears in Wilson's Theorem
- The Stirling approximation is for large values of this function
- The infinite series of the reciprocals of this function converge to e

Pi

- Ratio of a circle's circumference to its diameter; approximately equal to 3.14
- Archimedes bounded this number above and below using a 96-gon
- Buffon's needle was used to approximate this number
- The infinite series one over n-squared sums to this number divided by six
- This number is involved in the solution to the Basel problem
- This number being transcendental proved squaring the circle is impossible
- The gamma function evaluated at one-half yields the square root of this number

e

- Base of the natural logarithm; approximately equal to 2.72
- The infinite series one over n-factorial sums to this number
- As n approaches infinity, $(1+\frac{1}{n})^n$ approaches this number
- This number is raised to the negative one-half x-squared power in the pdf of the standard normal distribution
- This number raised to the x power is a function that is its own derivative

Sine

- Trig ratio of the opposite side to the hypotenuse
- Reciprocal is cosecant
- A law named for this function concerns the ratio of this function of a given angle to the side opposite it
- Derivative is cosine
- Gives the y-coordinate on the unit circle
- In Euler's formula, this function appears in the imaginary part [$e^{ix} = \cos(x) + i\ sin(x)$]
- Taylor series is $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

Cosine

- Trig ratio of the adjacent side to the hypotenuse
- Reciprocal is secant
- A law named for this function is a generalization of the Pythagorean Theorem
- Derivative is negative sine
- Gives the x-coordinate on the unit circle
- The hyperbolic form of this function is used to model the catenary (hanging chains, etc.)
- The Dottie number is a fixed point for this function equal to about 0.74
- In Euler's formula, this function appears in the real part [$e^{ix} = \cos(x) + i\ sin(x)$]
- The hyperbolic form of this function equals $\frac{e^x + e^{-x}}{2}$
- Taylor series is $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

Tangent

- Trig ratio of the opposite side to the adjacent side
- Equal to sine divided by cosine
- This function squared plus one equals secant squared
- Derivative is secant squared
- Taylor series is $x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$

Golden Ratio

- Denoted by the Greek letter phi
- Equal to $\frac{1+\sqrt{5}}{2} \approx 1.618$
- The limit of the ratio of consecutive Fibonacci numbers approaches this
- This number has a continued fraction consisting of entirely of ones
- Equal to $2\cos(36°)\ or\ 2\cos\left(\frac{\pi}{5}\right)$
- Equal to the ratio of the diagonal of a regular pentagon to its side length
- This value can be found in Binet's formula (CAREFUL with wording)
- One plus the reciprocal of this value is equal to this value
- The square of this value is equal to this value plus one

Greatest Common Divisor/Factor

- Defined as the largest number that will divide two integers
- This operation can be performed using the Euclidean algorithm
- This operation is the subject of **Bezout's Identity**
- If this operation yields one, the two inputs are relatively prime

Fractals

- Objects that consist of self-similar patterns
- Include Mandlebrot Set, Sierpinski Triangle/Gasket/Carpet, Koch Snowflake, Menger Sponge, Cantor Set, Julia Set, and space-filling curves
- First studied by Benoit Mandlebrot
- Have a topological dimension that is always smaller than the Hausdorff dimension
- Can be generated by L-system and strange attractors

Triangles

- The area of these can be calculated using the semiperimeter via Hero(n)'s Formula
- Euler names a line passing through three points in these (the orthocenter, circumcenter, and centroid)
- The angles of these when inscribed in a circle is described by Thales' Theorem

Circles

- Conic section with zero eccentricity
- Defined as the set of all points exactly equidistant from another point (the single focus)
- This figure maximizes the area to perimeter ratio
- Parametrically defined as r equals a constant
- Squaring this shape was shown to be impossible when pi was proven to be transcendental
- The nine-point one of these intersects the midpoints of all the sides and the feet of all the altitudes of a triangle
- The "great" ones of these are the equivalent of lines in spherical geometry
- A torus is the Cartesian product of two of these

Hyperbolas

- Conic section that has two non-connected branches
- Has an eccentricity greater than one
- Exemplified by $y = \frac{1}{x}$
- The difference of the distance of any point in this figure to its two foci is constant
- Apollonius of Perga showed that an angle could be trisected using these
- The wave equation is this type of PDE (partial differential equation)
- Unbound comets travel along a path of this shape

Ellipses

- Conic section resembling an elongated circle
- Has an eccentricity between zero and one
- The sum of the distance of any point in this figure to its two foci is constant
- The area of this figure is pi times the product of the semi-major and semi-minor axes
- Parametrically defined as $x = a\cos(t)$ and $y = b\sin(t)$ with a and b not necessarily equal
- Drawn by the trammel of Archimedes
- Lame [la-MEY] curves are known as the "super" type of these
- Curves named for these are modular according to the Taniyama-Shimura Conjecture

Standard Deviation

- Symbolized lower case sigma for populations and S for samples
- Measure of spread of a distribution that is equal to the square root of the variance
- The 68-95-99.7 rule (empirical rule) concerns being this many values away from the mean
- Z-scores are expressed in units of this value from the mean
- This value is the subject of Chebyshev's Inequality
- The correlation of variables X and Y is equal to the covariance of X and Y divided by the product of this value for X and this value for Y

Variance

- Symbolized lower case sigma-squared for populations and S^2 for samples
- Measure of spread of a distribution that is equal to the square of the standard deviation
- This value equals the expected value of X^2 minus the quantity, the expected value of X, end quantity, squared OR $E(X^2) - E(X)^2$
- For binomial distributions, this value equals $np(1-p)$ where n is the sample size and p is the probability of success
- For Poisson distributions, this value equals the expected value (mean)

Triangle Inequality

- States that the sum of the lengths of two sides of a triangle must be larger than the third side
- Can be restated as $|x + y| \le |x| + |y|$
- Is a special case of the Cauchy-Schwarz Inequality
- Minkowski generalized this statement to L-p spaces
- This statement is reversed in Minkowski spaces
- Can be used to show the packing number is less than or equal to the intrinsic covering number

Commutative Property

- States that changing the order of operands does not matter (for addition, a + b = b + a)
- Abelian groups have an operation with this property
- Matrix multiplication and function composition do not have this property (nor do division and subtraction)

Fundamental Theorem of Algebra

- States every non-constant polynomial has at least one complex root (OR every polynomial of degree n has n complex roots if multiplicity is counted)
- Can be proven using Liouville's Theorem

Fundamental Theorem of Arithmetic

- States every integer greater than one has a unique prime factorization

Four Color Map Theorem

- Provides a limit to the number of hues needed to create a map
- Originally solved by having a computer check 1,936 cases
- Haken and Appel proved this theorem
- Shows that the Heawood conjecture holds for everything but Klein bottles
- Originally known as Guthrie's conjecture
- Led to the development of Kempe chains

Chinese Remainder Theorem

- First proposed by Sun Tzu
- Supposedly used to count the number of soldiers in an army
- Used to solve systems of congruences where the moduli are pairwise coprime
- Asmuth-Bloom secret sharing scheme relies on this theorem
- Mignotte secret sharing scheme relies on this theorem
- This theorem is used by the Good-Thomas algorithm
- An extension of this theorem can be used to show an isomorphism between a quotient ring and a product of quotient rings

Eigenvalues

- These numbers, represented lambda, are the zero of the characteristic polynomial (values of λ that make $|A\lambda - I| = 0$)
- The product of these numbers always equals the determinant
- The sum of these numbers always equals the trace